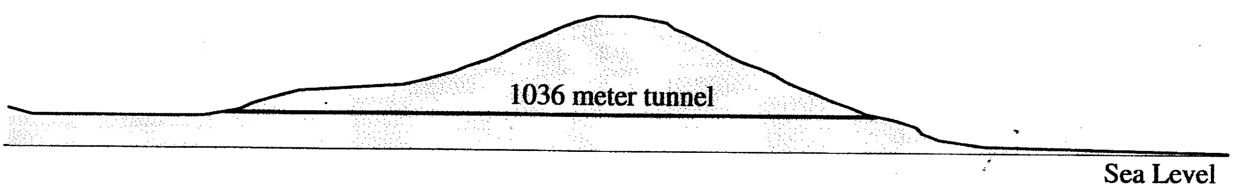
Project MATHEMATICS!

Program Guide and Workbook

to accompany the videotape on

THE TUNNEL OF SAMOS



Project MATHEMATICS!
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, CA 91125 • (818) 395-3759

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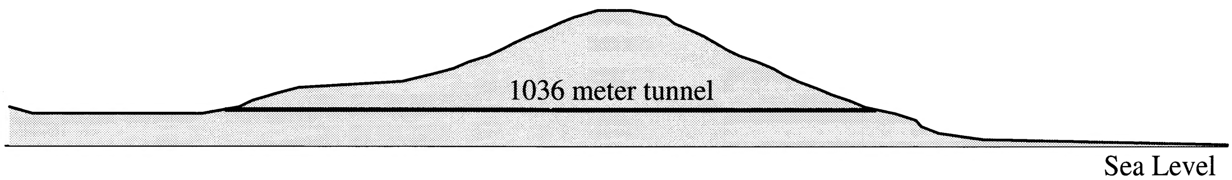

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THE TUNNEL OF SAMOS

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AIMS AND GOALS OF *Project MATHEMATICS!*

Project MATHEMATICS! produces computer-animated videotapes to show students that learning mathematics can be exciting and intellectually rewarding. The videotapes treat mathematical concepts in ways that cannot be done at the chalkboard or in a textbook. They provide an audiovisual resource to be used together with textbooks and classroom instruction. Each videotape is accompanied by a workbook designed to help instructors integrate the videotape with traditional classroom activities. Video makes it possible to transmit a large amount of information in a relatively short time. Consequently, it is not expected that all students will understand and absorb all the information in one viewing. The viewer is encouraged to take advantage of video technology that makes it possible to stop the tape and repeat portions as needed.

The manner in which the videotape is used in the classroom will depend on the ability and background of the students and on the extent of teacher involvement. Some students will be able to watch the tape and learn much of the material without the help of an instructor. However, most students cannot learn mathematics by simply watching television any more than they can by simply listening to a classroom lecture or reading a textbook. For them, interaction with a teacher is essential to learning. The videotapes and workbooks are designed to stimulate discussion and encourage such interaction.

STRUCTURE OF THE WORKBOOK

The workbook begins with a brief outline of the video program, followed by suggestions of what the teacher can do before showing the tape. Numbered sections of the workbook correspond to capsule subdivisions in the tape. Each section summarizes the important points in the capsule. Exercises are included to help strengthen understanding. The exercises emphasize key ideas, words and phrases, as well as applications. Some sections suggest projects that students can do for themselves.

I. BRIEF OUTLINE OF THE PROGRAM

The videotape begins with a brief *Review of Prerequisites*, dealing with a property of similar triangles introduced in a previous module on *Similarity*. The *Tunnel of Samos* module tells the story of one of the greatest engineering feats of the ancient world.

The water supply of the principal city on the island of Samos in ancient Greece was inadequate for its growing population, but there was an ample supply in the mountains. To bring water from the mountains to the city, a one-kilometer tunnel was dug in the 6th century B.C. through a large hill of solid limestone. The tunnelers worked from both ends and met in the middle, more or less as planned. This module shows how similar triangles probably were used to determine the correct direction for tunneling. The workers who carved the tunnel with primitive tools met at the center with an error less than 0.15% of the length, a remarkable achievement for that era. The module explains why some error could be expected. It also shows that the problem of delivering fresh water to large populations has been an ongoing human endeavor since ancient times.

After centuries of neglect the tunnel became lost until it was rediscovered in 1882 in a relatively good state of preservation. It contained artifacts dating back to the Roman and Byzantine eras. Shortly thereafter the German archaeologist Ernst Fabricius surveyed the tunnel and published a full description.

II. BEFORE WATCHING THE VIDEOTAPE

This program builds on a few mathematical ideas students should be familiar with. They are included in the following list of key words and statements.

KEY WORDS AND STATEMENTS:

Hypotenuse and *legs* of a *right triangle.*

A *change of scale* transforms a figure into a similar one of the same shape but different size.

Expanding or *contracting* a triangle by a scaling factor changes the lengths of its sides, but does not change the angles or the ratios of lengths of *corresponding sides.*

CHALLENGING PROBLEM:

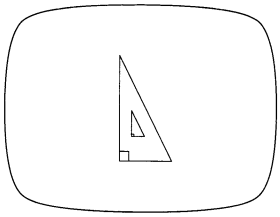
The program focuses on the following question, which arose from necessity in ancient Greece:

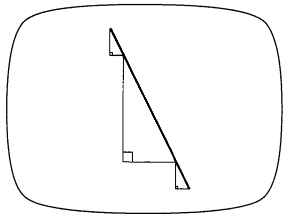
How can two crews dig a straight level tunnel through a mountain, working from opposite ends, and be sure of meeting in the middle?

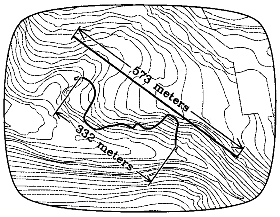
Today, with the use of aerial photography, radio signals, topographic maps, surveying instruments, and laser-controlled tunneling equipment, it is relatively easy to plan the correct direction for tunneling, but without this technology the solution of the problem is not obvious. How would you do it?

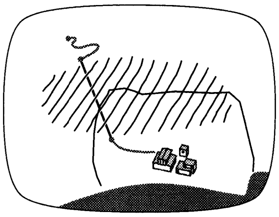
Before reading this booklet or viewing the videotape to learn how the ancient Greeks might have answered this question, you may wish to spend some time thinking about how you would answer the question by yourself, without using the tools of modern technology. This is time well spent, even if you don't find a solution, because it will give you a better appreciation of the ingenuity and skill that was required to provide a solution.

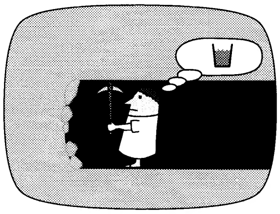
III. TABLE OF CONTENTS

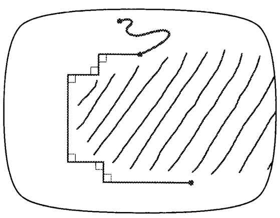


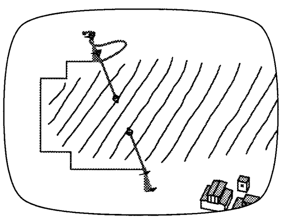












Review of Prerequisites

Figure 1 shows two similar triangles. The triangles have the same shape because corresponding angles are equal. But the triangles have different sizes, the edges of the larger one being twice as long as those of the smaller. We label the lengths of the edges of the smaller triangle a, b, c and the corresponding lengths in the larger triangle a', b', c'. In the two triangles, b and b' are the shortest sides, c and c' are the longest. The sides of the larger triangle are twice as long as those of the smaller. In symbols, $a' = 2a$, $b' = 2b$ and $c' = 2c$, from which we find that lengths of corresponding sides have the same ratio:

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = 2.$$

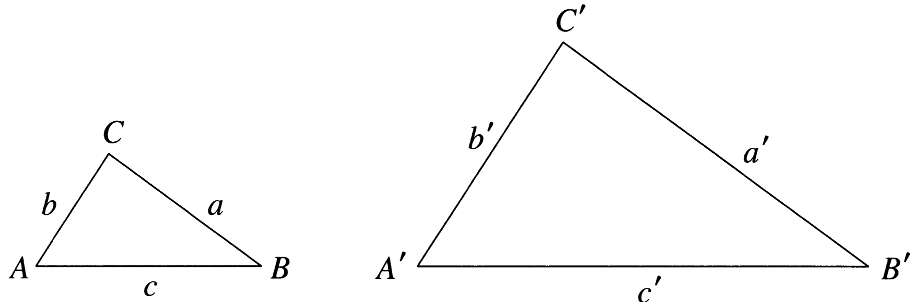


Figure 1. Similar triangles.

Any two triangles whose corresponding angles are equal are called *similar*, and we say they have the same shape. But, as in the example shown in Figure 1, similar triangles can be of different sizes. It can be shown that lengths of corresponding sides of similar triangles have the same ratio, a number that is called the *scaling factor*, or the *similarity ratio*. A scaling factor $s > 0$ is called an *expansion factor* if s is greater than 1, and a *contraction factor* if s is less than 1. If the scaling factor is equal to 1, the triangles are congruent.

In Figure 1 the sides of the larger triangle are twice as long as those of the smaller triangle, and we say that the larger triangle is obtained from the smaller by expansion by the factor $s = 2$. On the other hand, the smaller triangle is obtained from the larger by contraction by the factor $s = 1/2$.

Conversely, if the lengths of corresponding sides of two triangles ABC and $A'B'C'$ have the same ratio it can be shown that corresponding angles are equal (have the same measure),

$$\angle A = \angle A', \ \angle B = \angle B', \ \angle C = \angle C'.$$

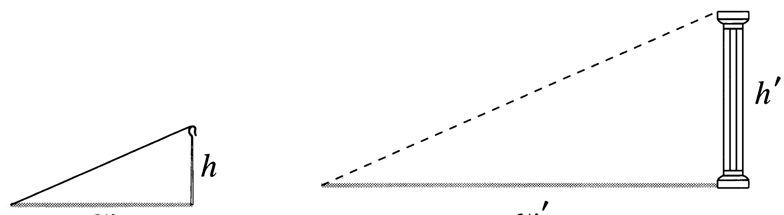
In other words, expansion or contraction of a triangle by a scaling factor does not change the angles.

More information about the concept of similarity can be found in the workbook for the module entitled *Similarity*. Some exercises from that workbook are included in an appendix near the end of this booklet.

One particular application of similarity is relevant to our story because it is attributed to Thales, who lived in the sixth century B.C. in Miletus on the coast of Asia Minor opposite the island of Samos. (See

the map on page 10.) Thales had traveled to Egypt and brought back many empirical rules or recipes for computing quantities related to land measurement. By trying to fit these recipes into a logically connected system, Thales began the Greek tradition of using logical reasoning to deduce properties of geometric figures. He is often credited with constructing a framework for geometry which became the foundation for the deductive system organized and expounded so well two centuries later by Euclid in his *Elements*.

One of the earliest applications of similarity was to determine the height of a tall object, such as a tree or column, without measuring its length directly. Thales invented a method for determining the height of a column by comparing the length of its shadow with that of his staff. Figure 2 shows a column whose height h' is to be determined.



Shadow cast by Thales's staff

Shadow cast by a Greek column

Figure 2. The method of Thales for determining the height of a column.

The quantities that are known or can be measured are the height of Thales's staff, h, and the lengths of the two shadows, w and w'. A line from the end of the shadow of Thales's staff to the top of the staff is the hypotenuse of a right triangle. The dotted line from the end of the shadow of the column to the top of the column is the hypotenuse of another right triangle. If the shadows are cast at the same time of day, these two lines, which represent the direction of the sun's rays, can be regarded as parallel because the sun is so far away. Therefore the two triangles are similar because corresponding angles are equal. Consequently, corresponding sides have the same ratio: $h'/h = w'/w$. Solving this equation for h' we find

$$h' = h\left(\frac{w'}{w}\right).$$

In other words, the height of the column, h', is equal to the height of the staff, h, multiplied by the ratio of the lengths of the shadows.

This simple solution can be simplified even further if the shadows are measured at the time of day when the shadow length w is exactly equal to the height h of the staff. When $w = h$, the equation $h' = hw'/w$ simplifies to $h' = w'$. It seems likely that Thales would have used this simplified method because no calculations are needed. The arithmetical operations of determining the ratio of shadow lengths, and multiplying this ratio by h were not part of the mathematical tools available in the sixth century B. C. Although these calculations are routine in the modern world, it should be realized that the introduction of algebraic notation, Arabic numerals, decimal notation, and their use in doing simple arithmetical operations such as multiplication and division, came into being many centuries after the time of Thales.

Because Miletus and Samos are near each other, the idea of using similar triangles for determining distances could very well have found its way to Samos.

1. *The quest for water*

Water is as essential to life as air. Extensive networks of irrigation and fresh-water canals as well as aqueducts and tunnels have existed since the beginnings of ancient communities. Even today water is carried a long way to thirsty cities like Los Angeles where there is little rainfall. And cities with more rainfall, like New York and Rome, also need aqueducts to bring water to their huge populations.

Sources of good drinking water are usually found in the mountains where the water has not yet been polluted by silt or impurities, whereas the major consumers are situated some distance away at lower elevations. When water is transported to a distant city by gravity the conduit must be sloped to permit downhill flow. If the conduit slopes only slightly the water is liable to stagnate and become unhealthful or to be lost by evaporation. But if it slopes too much the rushing water will cause erosion and may even destroy the channel carrying it. Slope requirements often result in lengthy meandering canals or aqueducts that are expensive to maintain and to defend against attack or sabotage. When hills intervene between the source and the terminus, it is often advantageous to tunnel through the obstacle, even though tunneling is more difficult and often more expensive than building a canal or aqueduct. Because the very life of a community depends on a steady supply of fresh unpolluted drinkable water, huge sums are often expended to deliver water to where it is needed.

This module describes a famous water tunnel 1,036 meters long excavated through a mountain on the Greek island of Samos in the sixth century B.C. The historian Herodotus regarded the tunnel of Samos (indicated by the rough sketch in Figure 3) as one of the three greatest engineering feats of ancient times. The other two were also on Samos: a mole or breakwater a quarter of a mile long forming an artifical harbor, and the temple of Hera, one of the largest temples ever built by the ancient Greeks.

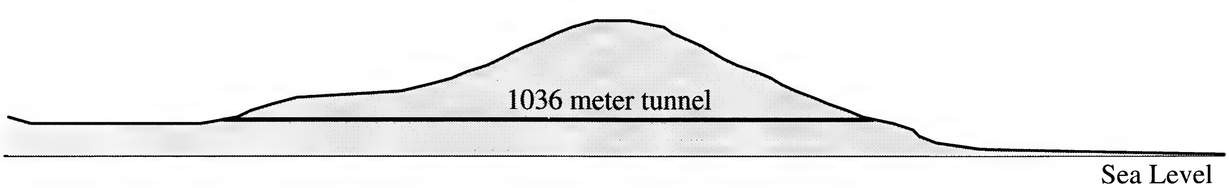


Figure 3. Profile sketch showing the approximate position of the tunnel of Samos through the mountain.

The tunnel of Samos was excavated by two teams advancing in a straight line from both ends. This raises an interesting intellectual question:

What method did they use to determine the correct direction for tunneling?

Two possible methods are described in this module. The first was proposed by Hero of Alexandria, five centuries after the tunnel was completed. Hero's explanation was widely accepted for nearly 2,000 years before modern scholars began to question Hero's theory. Two British historians of science visited the tunnel three decades ago to check the plausibility and practicability of Hero's solution. They raised serious questions and suggested an alternate method. The author of this booklet also visited the site in 1993 and gives his own reasons for concluding that a combination of the two methods may have been used.

The tunnel of Hezekiah

The tunnel of Samos was not the first to be excavated from both ends. Nearly two centuries earlier King Hezekiah of Judah ordered a tunnel carved through a third of a mile of solid rock to bring water from the spring of Gihon, a perennial source of water in the Ophel Hill, to a Jerusalem reservoir called the Pool of Siloam (Figure 4). Hezekiah's tunnel is famous for two reasons: first, for the debate concerning the method of excavating it; and second, for the six-line inscription on a stone table commemorating completion of the project around 700 B.C. This tablet contains the oldest known cursive writing in the Phoenician-Hebraic alphabet (Figure 5). Although the straight line distance from spring to pool is only 332 meters, the tunnel itself has a length of 573 meters. Its narrow width varies only slightly, from .58 to .65 meters, but its height is extremely irregular, ranging from 1.5 to 5 meters.

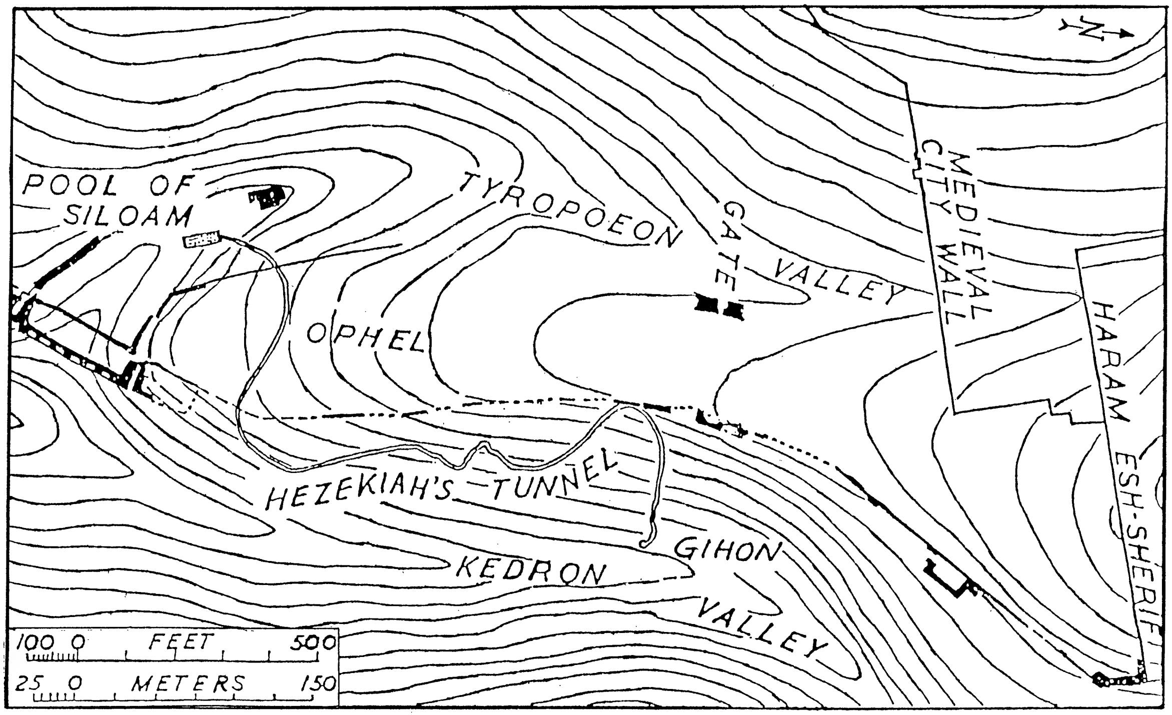


Figure 4. Contour map showing the sinuous path of Hezekiah's tunnel.

The tunnel's winding course, the variability of its height, and the manner in which separate teams of tunnelers managed to meet have been the subject of considerable debate. Some thought the tunnelers followed a relatively soft, easily quarried stratum in the bedrock. Others argued that the tunnelers continually changed course to avoid disturbing the tombs of the Davidic dynasty. A 1982 report of a hydro-geological survey concludes that the tunnel is most probably an enlargement of a preexisting natural conduit that originally carried water *toward* the spring. The variable height of the tunnel reflects adjustments made to guarantee the direction of flow from spring to reservoir. The meeting of the two teams of tunnelers was thus ensured because they followed the path of a preexisting channel.

Anticipating a siege by the Assyrian army, Hezekiah created this tunnel to divert the waters of the Gihon spring outside the city walls of Jerusalem to a protected reservoir. Although much of Judah was devastated, a secure water supply was a major element of Hezekiah's fortification that saved Jerusalem.

The inscribed tablet commemorating the triumphal meeting of the two tunneling crews was found on the east wall ten meters inside the southern end of the tunnel. The tablet was removed in 1890 and is now in the Imperial Museum in Istanbul. It reads as follows:

> This is the story of the boring through: whilst [the tunnellers lifted] the pick each towards his fellow and whilst three cubits [yet remained] to be bored [there was heard] the voice of a man calling his fellow, for there was a split in the rock on the right hand and on [the left hand]. And on the day of the boring through, the tunnellers struck, each in the direction of his fellows, pick against pick. And the water started to flow from the source to the pool, twelve hundred cubits. A hundred cubits was the height of the rock above the head of the tunnellers.

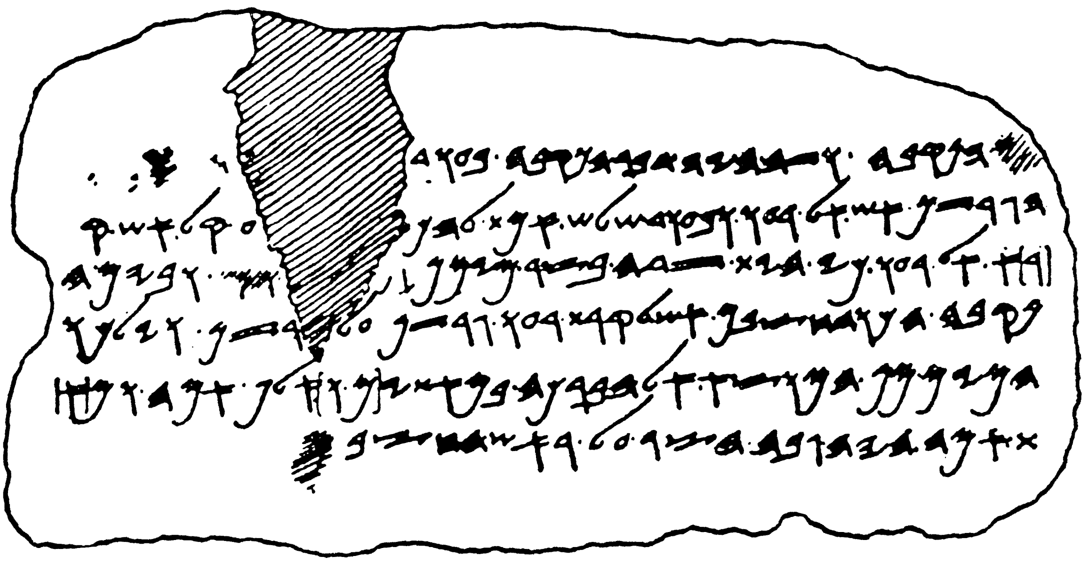
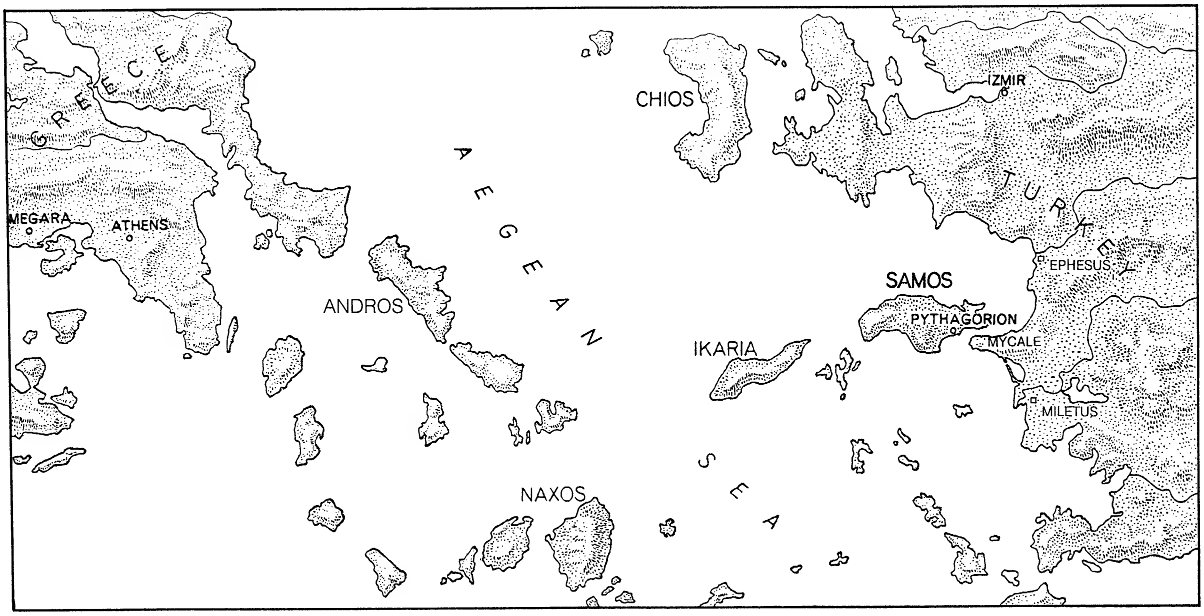


Figure 5. Cursive Phoenician-Hebraic commemorating the breakthrough in the tunnel of Hezekiah.

Even today, water flows through the tunnel of Hezekiah to the Pool of Siloam. The successful completion of the tunnel without using modern aids such as the magnetic compass or surveying instruments must be considered a remarkable engineering achievement for that period. The satisfaction indicated by the inscription may also reflect the realization that if the crews had not met, the engineer's life would doubtless have been forfeit. The inscription occupied the lower half of a prepared surface; it is probable that the upper half was intended for a relief showing the shape of the tunnel, with possibly the names of local political leaders inscribed thereon. But there is no written record naming the actual engineers, just as there is none on the pyramids of Egypt, most of the cathedrals of Europe, or most dams and bridges of the modern world.

The first hydraulic engineer whose name has been preserved is Eupalinos of Megara, a remarkable Greek engineer, who excavated a much longer tunnel straight through a mountain of limestone rock to bring water to the capital of ancient Samos. We return to the story of that achievement.

2. *Bringing water to Samos*



Samos, just off the coast of Turkey in the Aegean sea, is the eighth largest Greek island, with an area less than 200 square miles. It is separated from Asia Minor by the Strait of Mycale, a little more than a mile wide. Phoenician seafarers gave the island its name because of its high mountains, one of the striking features of the island. ''Sama'' in the ancient languages of the area meant ''a high place.'' It is a colorful island with lush vegetation, beautiful bays, and an abundance of good spring water.

Samos was colonized by Ionian Greeks around 1,000 B.C. and began to flourish during the seventh and sixth centuries B.C. It reached its greatest period during the reign of the tyrant Polycrates (570-522 B.C.), whose court attracted poets, artists, musicians, philosophers and mathematicians from all over the Greek world. His capital city, also named Samos, was situated on the slopes of a mountain (later called Mount Castro) dominating the natural harbor below and the narrow strip of sea between Samos and Asia Minor. Herodotus describes Samos as the most famous city of its time. The site of the ancient city is partly occupied today by the seaside village of Pythagorion, named in honor of Pythagoras, the mathematician and philosopher who was born on Samos around 572 B.C.

Polycrates had a stranglehold on all coastal trade that passed through the narrow Strait of Mycale. By 525 B.C. he was master of the eastern Aegean. His city was made virtually impregnable by a surrounding ring of fortifications that rose over the top of the 900-ft Mount Castro. The massive walls had an overall length of 3.9 miles and are among the best preserved in Greece. He built a huge breakwater to form an artificial harbor protecting his ships from the southeast wind. To secure protection in the afterlife he constructed a magnificent temple to Hera. And to provide his city with a secure water supply he carved a one-kilometer tunnel, two meters wide and two meters high straight through the heart of Mount Castro.

As the capital city flourished in the sixth century B.C. more fresh water was needed for its growing population. There was a copious spring at the site of a village (now known as Agiades) in a fertile valley northwest of the city, but access was blocked by Mount Castro. The water could have been brought around the mountain by an aqueduct, as the Romans were to do much later from a different source. Aware of the dangers of having a water course exposed to an enemy for even part of its length, Polycrates, like Hezekiah of Judah, ordered a delivery system that was to be completely subterranean.

A remarkable Greek engineer, Eupalinos of Megara, designed an ingenious system to meet Polycrates' requirements. Its principal feature was the tunnel through Mount Castro. The water was brought from its source at Agiades to the northern mouth of the tunnel by an underground conduit that followed an 850-meter sinuous course along the contours of the valley, passing under three creek beds enroute.

Today the hamlet of Agiades consists of a few houses and three chapels, all clustered around the generous spring. Water from the spring flows directly into an ancient reservoir beneath one of the chapels. This reservoir, undoubtedly built by Eupalinos, was roofed over to minimize evaporation and was camouflaged with earth. From the reservoir the water enters an underground passage just high enough and wide enough for a man to walk through. Probably this passage originally was dug as an open trench and later covered. The water was carried in round clay pipes set in the floor, and the passage was marked by many inspection shafts that still open to the surface. Below Agiades the passage travels south for about 120 meters, until it goes beneath the bed of a stream that flows only after a rain. It follows the contours for another 120 meters or so then bears east parallel to a larger stream for about 300 meters (see Figure 6). It later makes a U-turn under this stream and runs west along the northern slope of Mount Castro passing under another small and usually dry creek bed until it reaches the east wall of the tunnel, which it enters at a depth of approximately 3 meters below the tunnel floor. Inside the tunnel, whose floor is level, the water was carried in a sloping rectangular channel excavated below the floor on the eastern edge.

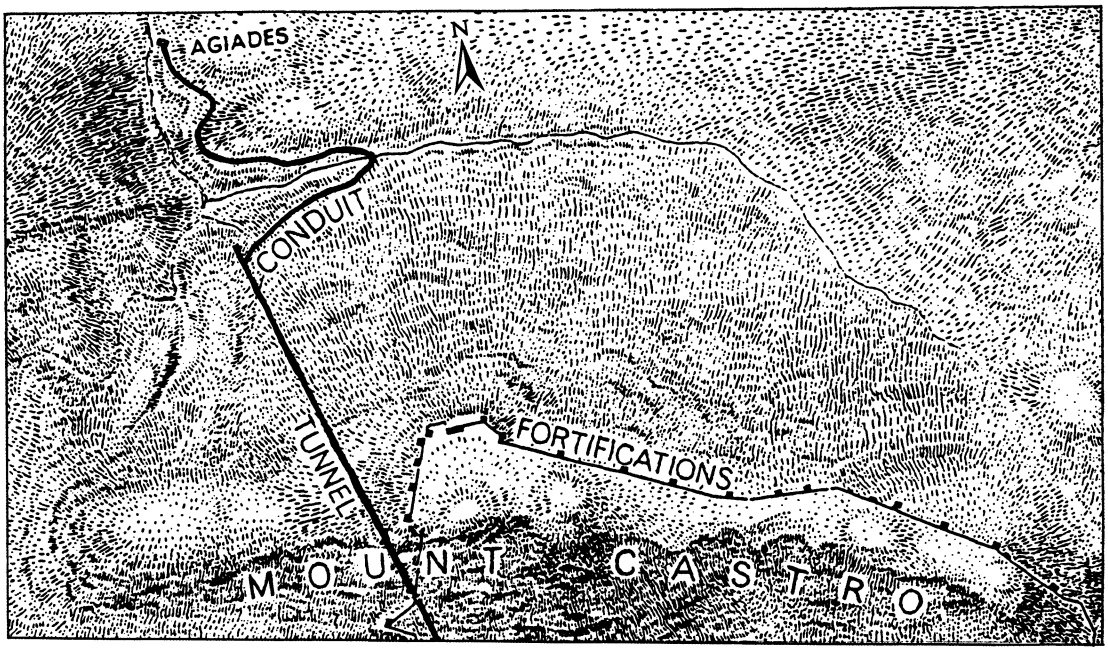


Figure 6. Upper portion of map shows the underground conduit from Agiades to north end of the tunnel.

In 522 B.C. Polycrates was tricked and captured by the Persians and crucified on the mainland opposite. Samos remained a strong power and played an important role in the victorious sea battle of Mycale against the Persians in 479 B.C. In 439 B.C. a large Athenian force led by Pericles captured Samos after a traitor revealed the secret of the underground conduit. With its water supply cut off, Samos had no choice but to surrender. As an ally of Athens during the Peloponnesian Wars, Samos was occupied by the Spartans and sacked. Samos later passed to Alexander the Great and his successors. In 129 B.C. Samos was conquered by Rome, and became the summer residence of emperors. Mark Antony and Cleopatra spent their honeymoon here. The Romans left artifacts in the tunnel but did not use it as a water channel; they built an aqueduct to bring water from another source to their baths on the city's shoreline.

3. *Excavating tunnels from both ends*

In modern times, digging a tunnel from both ends is almost a routine task, whether it's a few feet beneath the streets of Los Angeles, or a hundred meters below the surface of the English Channel. The first major railway tunnel to be excavated from both ends was planned by Germain Sommeiller, a young pioneer of Alpine tunneling from the kingdom of Sardinia, which then included part of northern Italy. In 1857, after nearly a decade of planning, the Sardinian government approved Sommeiller's design for driving a tunnel seven and a half miles long under Mt. Fréjus in the Alps. The Italian War of Liberation that broke out two years later did not interfere with the work. After the war, when Savoy was ceded to France in 1860, the French government agreed to share the costs of the tunnel. Four thousand workers using specially designed compressed-air drilling machines broke through at the junction in 1871, with an error in alignment of less than half a meter. Only fifty-five casualties were recorded, of which twenty-eight were fatal, a commendable safety record in that era for a massive project of that scale. The tunnel was officially opened on 17 September 1871 in the presence of 1,500 prominent guests invited from nearly every European country. Sommeiller was not present, having died from overwork in July of that year.

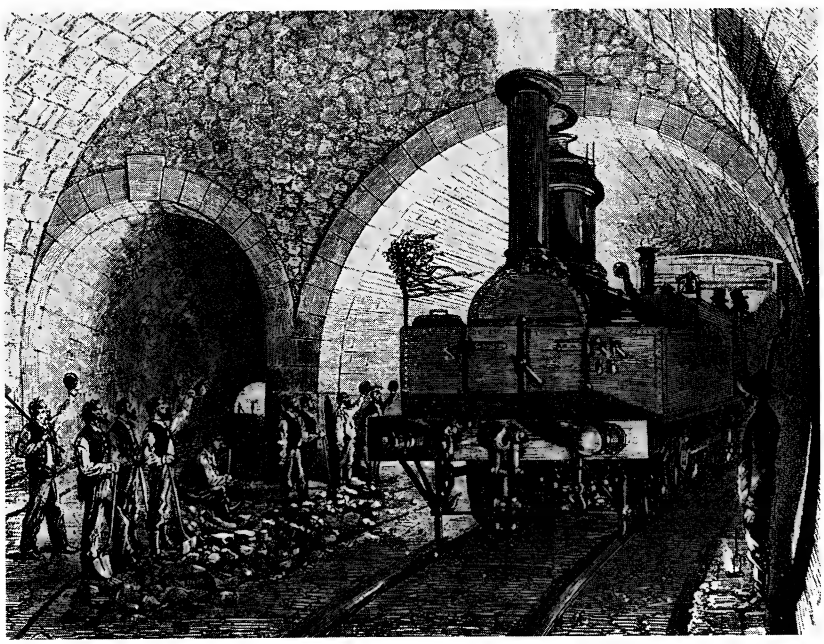


Figure 7. Inaugural train entering the first railway tunnel excavated from both ends.

The largest privately financed infrastructure of all times is the Channel Tunnel, commonly referred to as "The Chunnel," a 50-kilometer underwater rail link between England and continental Europe. It consists of a complex of three parallel tubes--two for trains moving in opposite directions, plus a service corridor--dipping as far as 45 meters below the sea floor of the English Channel. (See Figure 8.) It was dreamed of by Napoleon, futilely attempted 100 years ago, and completed in 1994 at a cost greater than most countries' gross national product. Like the tunnel of Samos excavated 2,500 years earlier, it was drilled from both ends. Fifteen thousand workers utilized all the power of modern technology, from computers to huge rotary machines that bored tubes 8.7 meters in diameter through the chalk substratum, advancing 45 meters per day.

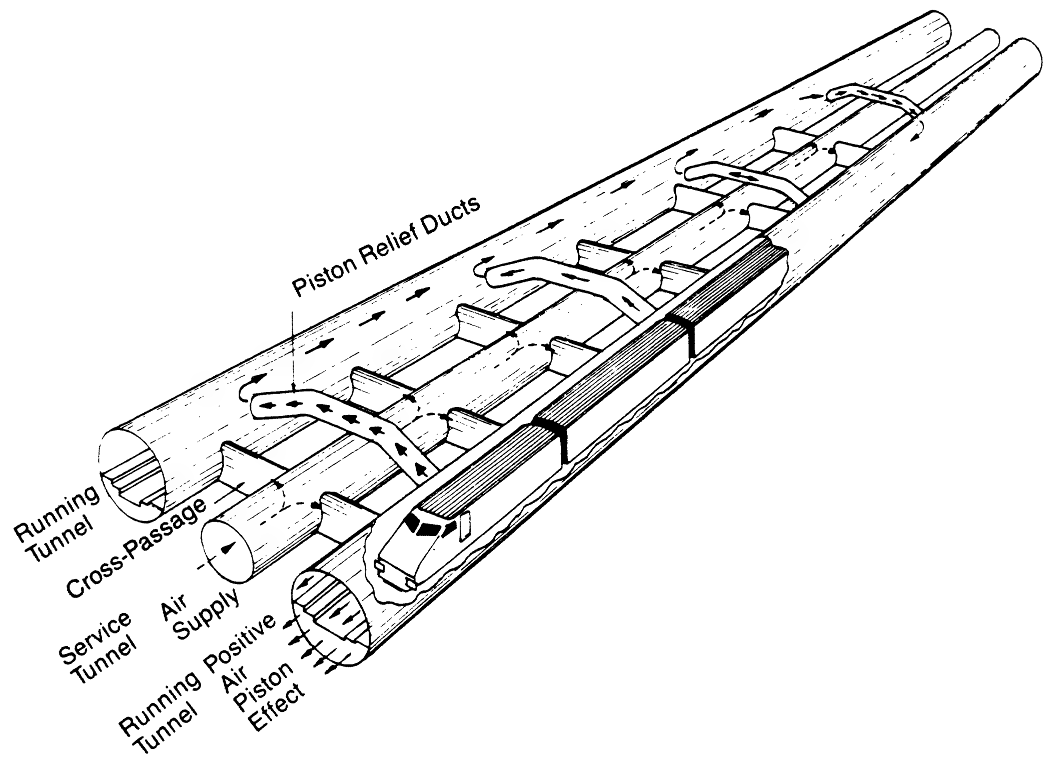


Figure 8. The tunnel under the English Channel was excavated from both ends.

By contrast, the tunnel of Samos was excavated by hand labor, with picks, hammers and chisels through more than a kilometer of solid limestone. This was a prodigious feat of manual labor, probably done by slaves. The intellectual feat of determining the direction of tunneling so the two crews would meet was equally impressive. How was it done? In the Iron Age the Greeks had no magnetic compass, no surveying instruments, and no topographic maps. In 530 B.C. they didn't even have much formally written mathematics at their disposal. Euclid's *Elements*, the first major compendium of ancient mathematics, was written in Alexandria some 200 years later. Although the mathematician Pythagoras was a Samian by birth, he spent little of his adult life in Samos, and there is no reason to believe that he or his mathematics played a role in designing the tunnel.

Pythagoras traveled extensively in Asia Minor and the Orient, and when he returned home to find Samos under the tyranny of Polycrates he decided to migrate to the Greek seaport of Crotona in Southern Italy, arriving there around 529 B.C. In Italy he founded the famous Pythagorean school, a closely-knit brotherhood with secret rites and observances, devoted to the study of philosophy, mathematics, and natural science. But the Greeks on Samos could have learned about right angles and similar triangles from Thales who lived on nearby Miletus on the mainland opposite. And most important of all, they had good common sense. Armed with these intellectual tools, Eupalinos pulled off one of the greatest engineering achievements of ancient times. No one knows for sure how he did it because no written evidence has come down to us from that era. But there are some convincing explanations, the oldest going back to Hero of Alexandria.

4. *Hero's explanation*

Hero of Alexandria was one of history's most ingenious scientific engineers. He flourished around 60 A.D. and founded the first organized school of engineering. The city of Alexandria had become the melting pot of antiquity, combining the intellectual heritage of the Greeks with the practical knowledge of the so-called barbarians. Hero produced a large technical encyclopedia describing early inventions, together with clever mathematical shortcuts. He was noted for thinking up charming theoretical solutions to difficult practical problems. One of his theoretical excursions, possibly inspired by the tunnel of Samos, was a method for aligning a level tunnel to be drilled through a mountain from both ends. It calls for making a series of right-angle traverses around the mountain, beginning at one entrance of the proposed tunnel and ending at the other. Figure 9 shows how the method could be applied to the terrain on Samos.

The dotted line labeled AC in Figure 9 denotes the tunnel direction to be determined. If Eupalinos laid his plans according to Hero's theory, he would start at a convenient point A near the northern entrance of the tunnel, and walk around the western face of the mountain along a series of right-angled traverses at a constant elevation above sea level, until he reached a point C near the southern entrance. By measuring the total distance moved west, and subtracting it from the total distance moved east, Eupalinos could determine one leg BC of a right triangle whose hypotenuse AC was along the proposed line of the tunnel. Then by adding north-south segments, he could also calculate the length of the other leg BA. Once Eupalinos knew the lengths of legs BA and BC, even though they were buried beneath the mountain, he could lay out smaller right triangles having the same shape as triangle CBA on the terrain to the north and to the south. For example, starting at A, he could mark off one-fifth of the east-west leg CB along a parallel direction to point D, then turn north at a right angle a distance one-fifth the south-north leg BA to point E, staying at the same elevation. The small right triangle ADE is similar to the large right triangle CBA, with each hypotenuse on the same line. Therefore, the north crew could always look back to markers along hypotenuse AE to make sure they were digging in the right direction. In the same way, markers could be located along the hypotenuse of a small right triangle visible from the south entrance to fix the direction for the other crew. And the distance AC would be five times that of hypotenuse EA. The completed level tunnel is about 55 meters above sea level, but the path used by Eupalinos may well have been nearer the 45 meter contour where the terrain is more easily traversed.

This remarkably simple and straightforward method requires that two independent tasks be carried out with great accuracy: (a) maintain a constant elevation while going around the mountain; and (b) determine a right angle when changing directions. Hero suggests that these early engineers could have done both with a *dioptra*, a primitive instrument used for leveling and for measuring right angles.

Hero's explanation, including the use of the dioptra, was widely accepted for nearly 2,000 years. In modern times the method was publicized in writings by distinguished science historians such as B. L. van der Waerden and Giorgio de Santillana. But some scholars have raised doubts that Eupalinos used Hero's method because there's no evidence to indicate that the dioptra existed as early as the sixth century B.C. Critics of Hero's explanation dismiss it out of hand because they invariably associate the method with a dioptra or similar surveying instrument. But there is another way to carry out Hero's method without a surveying instrument, especially if one separates the problems of right angles and leveling.

First, consider the problem of right angles. The Samians of that era certainly knew how to construct right angles with great accuracy, as evidenced by the huge rectangular stones in the beautifully preserved ancient walls that extended nearly four miles around the entire ancient capital of Samos. Dozens of right angles were also used in building the huge temple of Hera just a few miles away. We don't know exactly

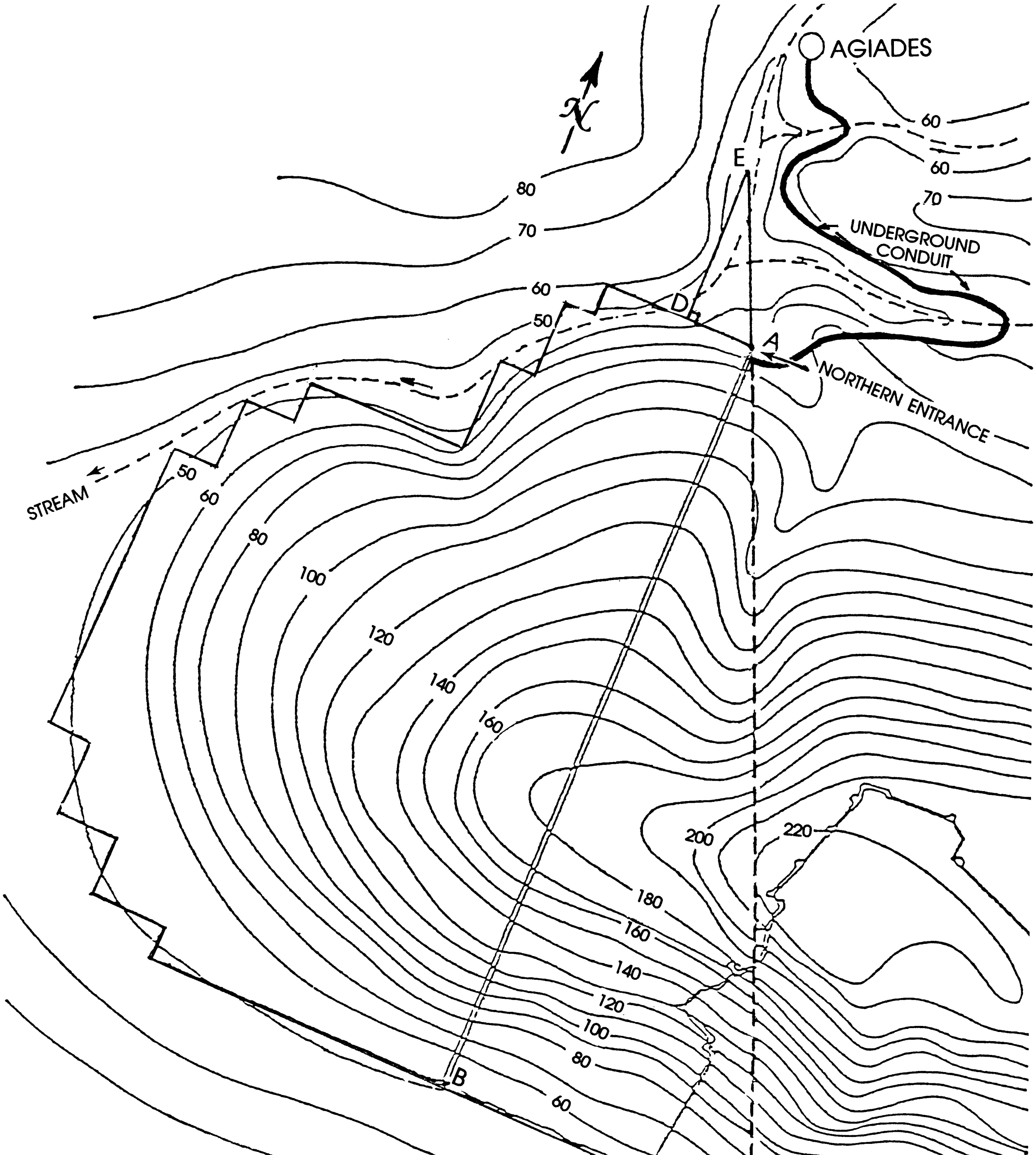
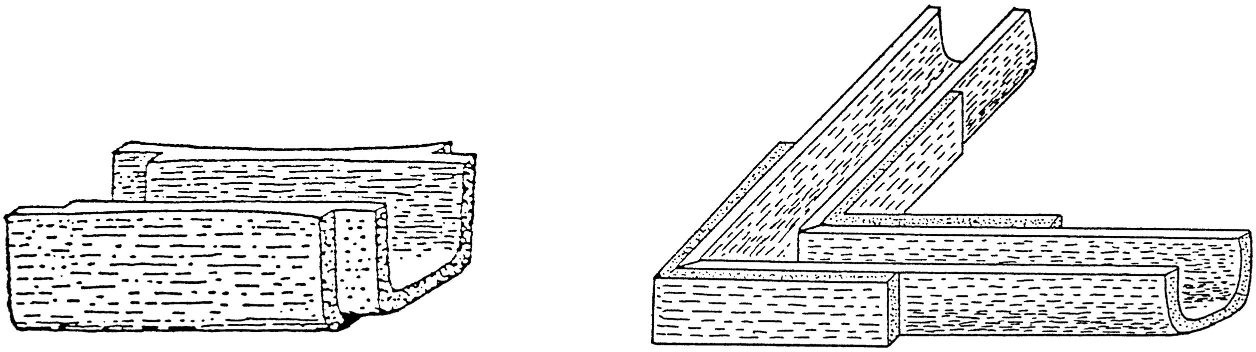


Figure 9. Right-angle traverses around Mount Castro joining two points near the tunnel openings.

how they determined right angles because no written records have survived. Possibly they constructed a portable rectangular frame with diagonals of equal length to ensure perpendicularity at the corners. Or they could have produced slabs of clay bent into L-shaped pieces to form right angles. In any case, it's reasonable to assume that constructing right angles with great accuracy was not a serious problem.

As for leveling, one of the architects of the temple of Hera was a Samian named Theodoros, who invented a primitive but accurate leveling instrument using water enclosed in a rectangular clay gutter. Beautifully designed round clay pipes from that era were found in the underground conduit outside the tunnel, and open rectangular clay gutters were found in the water channel inside the tunnel (Figure 10a). So the Samians had the capability to construct any number of clay gutters for leveling, and clay L-shaped pieces for joining the gutters at right angles, as suggested in Figure 10b. With an ample supply of limestone slabs available on Mount Castro and a few skilled stone masons, Eupalinos could have marked the path with a series of layered stone pillars capped by leveling gutters that maintained a constant elevation while going around the mountain. In fact, Eupalinos could have easily used clay gutters to construct a miniature constant-level aqueduct around the mountain joining the two tunnel entrances, thereby verifying constant elevation with considerable accuracy and avoiding the problem of measuring differences in elevation. Moreover, by using leveling gutters of a fixed length the net east-west and north-south distances could be accurately determined by counting the number of gutters and applying simple addition and subtraction. A temporary miniature aqueduct of right-angled traverses would be relatively easy to construct, especially if it followed a lower contour where the terrain is fairly smooth. Once it served its purpose it could be dismantled and the gutters recycled for later use inside the tunnel.



(a) (b)

Figure 10. (a) Clay gutters found inside the tunnel of Samos. (b) A way of joining them at right angles.

Note. Primitive water leveling devices are still used today to compare elevations of two points not visible from one another. For example, if there is reason to suspect that the ground beneath one corner of a building has settled, the difference in elevation between this corner and the one diagonally opposite can be determined by running a rubber or plastic hose around the building from one corner to the other. By filling the hose with water and placing glass tubes at each end, one can determine the difference in elevation between the two corners.

Suggested project:

Test Hero's theory on a level surface, such as a playground. To determine right angles, fasten four sticks together to form a rectangular frame with diagonals of equal length. Simulate the tunnel entrances by marking two points on the surface a convenient distance apart (say 50 feet, measured with a tape measure). Lay out a path consisting of several right-angle traverses analogous to that in Figure 9, and determine a large right triangle whose hypotenuse joins the two entrances. Lay out similar right triangles at each entrance with legs one-tenth those of the large right triangle. Use a long taut string to check if the three hypotenuses lie on a line.

5. *Another explanation*

Two British historians of science, June Goodfield and Stephen Toulmin, visited the site of the tunnel of Samos in 1958 and again in 1961 with the express purpose of checking the practicability and plausibility of Hero's theory of tunnel alignment as described in *Science Awakening*, B. L. van der Waerden's history of ancient mathematics. They spent several days studying the construction of the tunnel and the layout of the surrounding countryside with Hero's solution in mind. They believed that for practical reasons it would have been extremely laborious--if not actually impossible--to carry out Hero's solution, and they suggested an alternate method that involves going over the top of the mountain rather than around it.

Goodfield and Toulmin raise three basic questions about van der Waerden's account. (a) Does it fit, accurately and in detail, the *actual facts* about the tunnel? (b) Would it have been *possible* to construct this particular tunnel according to Hero's recipe? (c) Would it have been *necessary* to do so?

In response to (a) they reveal that van der Waerden is in error when he refers to a number of vertical shafts from the surface to the tunnel. In the tunnel itself, only one vertical shaft exists near the southern entrance. Other vertical shafts do exist, but they are outside the tunnel, joining the surface to the underground conduit. Although this error might cast some doubt on the accuracy of van der Waerden's description, it is really a minor point because the possibility of drilling vertical shafts joining the inside of the tunnel to the surface is not relevant to Hero's method of tunnel alignment.

Goodfield and Toulmin divide question (b) into two parts:

(1) Would Eupalinos have possessed instruments for taking the necessary vertical and horizontal sights with sufficient accuracy? There is general agreement among modern scholars that the dioptra probably was *not used.* In fact, as described in the foregoing section, an instrument like the dioptra was *not needed.*

(2) Would the nature of the surrounding terrain have permitted their use? In reply they write:

> "The moment one starts walking around on the site, the practical snags become apparent. If one is to construct the hypothetical right-angled plane triangle *inside* the mountain, one must keep on a constant horizontal contour *outside* it. This is next to impossible, as we found for ourselves. The western side of the mountain is extremely rough, being intersected with ravines....To survey a series of similar triangles in such territory would *in practice* have been a problem in three-dimensional geometry. Eupalinos would have needed to take some hundreds of vertical sightings, in addition to the horizontal ones, before he could reproduce on the ground the proportions of Hero's similar triangles with sufficient accuracy."

Armed with the knowledge of Goodfield and Toulmin's analysis, the author of this booklet also visited the tunnel site on Samos in the spring of 1993 to check for himself the feasibility of Hero's method. It is true, as the British historians reported, that the terrain following the 58-meter contour line from the southern entrance of the tunnel is quite rough, especially at the western face of the mountain. However, just a few meters below the south entrance, near the 45-meter contour, the ground is fairly smooth and it is easy to follow a goat trail through the brambles traversing the south face of the mountain. The terrain is such that Eupalinos could have cleared a suitable path and marked it with stone pillars, keeping them at a constant elevation with clay leveling gutters as described earlier. At the western end of the south face the terrain gradually slopes down into a stream bed which was nearly dry at the time the author visited the site. It is easy to walk along this stream bed along the western face of the mountain to reach the northern face which then gradually slopes upward toward the northern entrance of the tunnel.

Incidentally, the distances involved are not great. The total length of the path around the mountain from one entrance to the other is at most a mile and a half. The author and his wife, neither of whom is an expert hiker, traversed this path in about an hour and a half, following the contour map on page 15. Most of the time was spent searching for the location of the northern entrance, which was hidden by groves of pine trees. Contrary to the conclusions of Goodfield and Toulmin, our experience convinced us that there is no reason to dismiss Hero's solution as being impossible to carry out in practice.

Question (c) raised by Goodfield and Toulmin seems to be the most serious. They write:

> "...there is one curious point to be noticed about the precise location of the tunnel. It was driven, not through the center, but through the western end of Mount Castro; as a result, the surface conduit on the north side doubles back for several hundred yards along the hillside after crossing the stream, before it enters the mountain. Also on the southern side the surface conduit had to be correspondingly lengthened in order to get the water to the town....there is no reason why Eupalinos should not have driven through the mountain at any point he chose. By working a quarter of a mile further east, he could have shortened both surface conduits, and brought the water out much nearer to its destination. Why did he not do so?
>
> In the answer to this question, we suggest, lies the essential clue to Eupalinos' method. Having walked over Mount Castro several times by a variety of routes, we noticed one striking fact: the tunnel was built along one of the few lines by which one can climb easily and directly up the rugged southern hillside and then down the gentler northern slope to the valley behind. Specifically, the section of the hillside immediately behind the ancient town and harbor is far steeper and rougher than the section further west, through which the tunnel was actually driven.
>
> The relevance of this fact to Eupalinos' practical problem is as follows:...the most natural way to establish a line of constant direction across the mountain would have been to drive a line of posts into the ground, up one face of the hill, across the top and down the other....Along the actual line of the tunnel, there is in fact nothing to stop one aligning a series of posts by eye, to an overall accuracy of better than one degree....
>
> Eupalinos would, of course, have needed also to compare heights on the two sides of the hill....Having established the line of posts, one need only measure off the base of each post against that immediately below it, using a level; the rest is addition and subtraction."

The proposed solution of a row of posts going over the top of the mountain may sound reasonable, but it presents new problems. First, it would be difficult to drill holes in the rocky surface to install a large number of wooden posts. Instead of using wooden posts, it seems more realistic that Eupalinos would have built layered stone pillars or a low continuous wall (with only two or three layers of stone) to mark the path over the top. Stone was readily available on the site, and holes would not be needed.

Second, it is not trivial to align posts or pillars by eye on a hillside. An error of one degree in alignment could put the two crews nearly 10 meters apart at the proposed junction. Keeping track of differences of elevation without surveying instruments is even more difficult. And there is a greater chance of error in measuring many changes in elevation along the face of a hill than there would be in measuring horizontal distances going around on a path of constant elevation. Because errors can accumulate when making a large number of measurements, Eupalinos must have known that going over the top would not give a reliable comparison of altitudes of the two entrances. To insure success he knew it was essential for the two crews to dig along a level line joining the two entrances. The completed tunnel shows that he did indeed establish such a level line, so he must have used a method of leveling that left little margin for error. Water-leveling gutters made of clay were available and reliable, and it would have been relatively simple to build a miniature constant-level aqueduct in the manner described above to guarantee the accuracy required.

Possible solution using a combination of the two methods

It is the opinion of this author that Eupalinos may well have used a combination of the two proposed methods, proceeding as follows. First he constructed the underground conduit from Agiades to Mount Castro. For the north entrance to the tunnel he selected a point that could be easily reached from the city by going over the top of the mountain. For the south entrance he needed a point inside the city at exactly the same elevation as the north entrance. To locate such a point he first tried marking a path from the north entrance over the top and down the southern face with stone markers or perhaps by a low continuous stone wall. Because the entire path or wall would not be visible from any point on the mountain, Eupalinos must have realized that it would not be straight enough to fix the direction for tunneling and also would not guarantee a terminal point with the correct elevation. So, using the reliable water-level method described above, he made right-angle traverses around the mountain along a level miniature aqueduct, starting from the northern entrance and ending at a convenient point of his choice on the southern face in the vicinity of the low wall. This gave him exactly what he needed: two terminal points having very nearly the same elevation above sea level. Then by laying out similar right triangles at each entrance, as suggested by Hero, he obtained an accurate fix on the direction for tunneling.

6. *Completing the tunnel*

No matter how the tunnel was planned, the excavation itself was a remarkable accomplishment. Did the two crews meet as planned? Not quite. If the diggers had kept faith in geometry and continued along the two straight line paths on which they started they would have made a nearly perfect juncture. However, as Figure 11 shows, the path from the north deviates from a straight line. When the northern crew was nearly halfway to the junction point they started to zig-zag, first towards the left, then right for about a hundred meters, then a sharper correction towards the left for an equal distance, then right again for about fifty meters, then a left turn to the junction point.

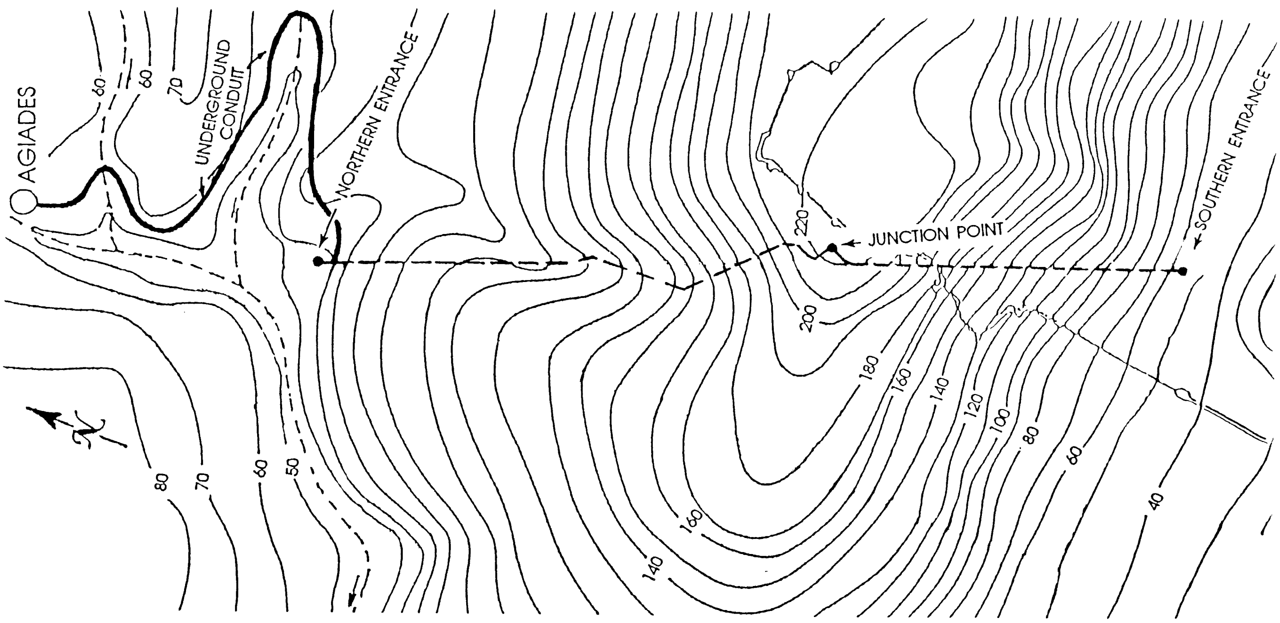


Figure 11. Plan view showing the approximate zig-zag course in the northern half of the tunnel.

Why did the northern crew decide to change course? No one knows for sure, but there are two theories. One is that zig-zagging was included as part of the master plan to avoid the possibility of digging two parallel shafts passing one another. If one shaft zig-zagged while the other continued in a straight line, intersection would be more likely. The other theory is that they changed direction to take advantage of the nature of the rock strata, detouring around pockets of soft material that would not support the ceiling, or places where water trickled through from the top. In the final stretch, when the two crews were near enough to begin hearing each other, both crews changed directions as needed and came together.

The sharp right-angle turn and the difference in floor levels at the junction prove conclusively that the tunnel was excavated from both ends. At the junction point itself the southern floor lies about 1½ meters below the northern floor, an error less than one-third of a percent of the distance excavated by each crew. This represents an engineering achievement of the first magnitude.

No one knows exactly how long it took to complete the project. Estimates range from 9 to 15 years. Working conditions during excavation must have been extremely unpleasant. Dust from the broken rock and smoke from oil lamps used for illumination would have presented serious ventilation problems, especially when the workers had advanced a considerable distance from each entrance.

A visit to the tunnel today reveals its full magnificence. Except for some minor irregularities, the southern half is remarkably straight. The craftsmanship is truly impressive, both for its precision and its high quality. The tunnel's two-meter height and width allowed workers carrying rubble to pass those returning for more. The ceiling and walls are exposed naked rock that gives the appearance of having been peeled off in layers. Water drips through the ceiling at many places and trickles down the walls. Calcium carbonate, the stuff from which stalagmites and stalactites are made, forms a glossy translucent coating over most of the walls, but some of the original chisel marks are still visible. And the floor is remarkably level, dropping less than two meters from the northern end to the southern end, eighty percent of that being the step at the junction. This indicates that Eupalinos took great care to make sure the two entrances were at the same level. Of course, such a gentle slope in a kilometer of length is not adequate to deliver a useful supply of water.

The water itself was carried in a sloping rectangular channel excavated below the floor of the tunnel along its eastern edge (Figure 12). Excavating this inner channel with hand labor in solid rock was another remarkable achievement, considering the fact that the walls of the channel are barely wide enough for one person to stand in. Yet they are carved with great care, maintaining a constant width throughout. The bottom of the channel is about three meters below the tunnel floor at the northern end and it gradually slopes down to more than nine meters below at the southern end. The bottom of the channel was lined with clay rectangular gutters, open at the top like those in Figure 10a.

When the tunnel was rediscovered in the latter part of the nineteenth century and partially restored, many portions of the channel were found to be covered with stone slabs, located from two to three meters above the bottom of the channel, on which excess rubble from the excavation had been placed up to the tunnel floor as seen in Figure 12. The generous space along the entire bottom of the channel provided a conduit large enough for ample water to flow through and also permitted a person to enter for inspection or repairs. In several places the channel was completely exposed all the way up to the tunnel floor, permitting inspection of the flow without entering the channel. Today most of the rubble has been cleared from the southern half of the channel and metal grillwork has been installed to prevent visitors from falling in. (Figure 13.) A barrier north of the junction point prohibits visitors from going any further. Access north of the barrier is blocked by a stalactite wall and debris that has not yet been removed.

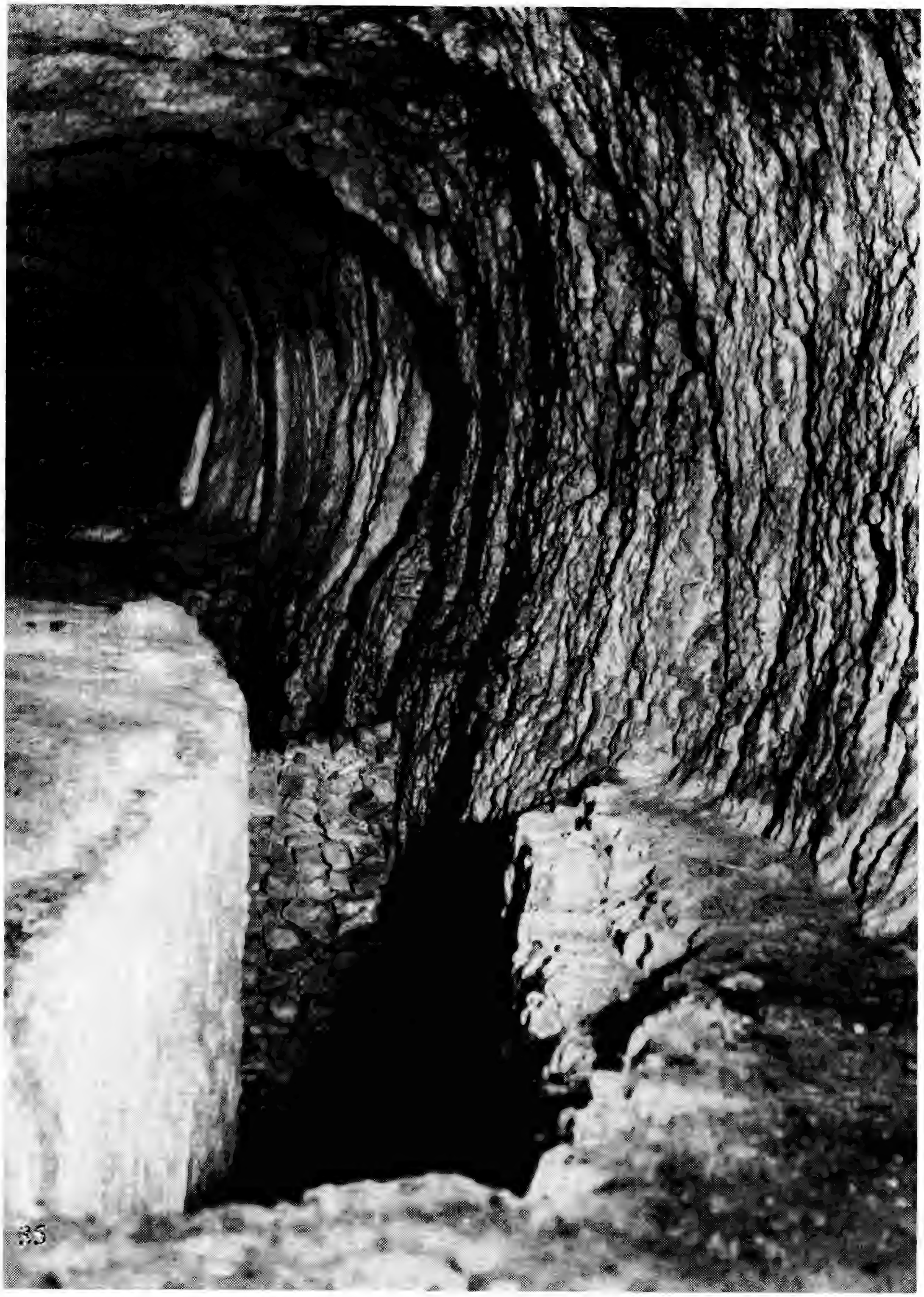


Figure 12. Tunnel interior, looking north. The water channel is along the eastern edge.

Topics for Discussion Concerning the Tunnel of Samos

1. Because water will not flow in a level tunnel, Eupalinos excavated a sloping channel alongside his level tunnel to carry the water from Agiades. If he had designed a sloping tunnel at the outset the inner channel would not have been needed. Why do you think he did not do so?

2. The vertical distance from the tunnel to the top of Mt. Castro is about 165 meters. Therefore, if a line of posts, each 1.5 meters high, was aligned by eye over the top of Mt. Castro from one tunnel entrance to the other, as suggested by Goodfield and Toulmin (p. 18), at least 220 posts would be needed. Determine the maximum error allowed in measuring the elevation of each post against that immediately below it to insure that the difference in elevation of the two entrances would be no more than 2 meters.

3. Why do you think the north crew decided to zig-zag long before reaching the half-way point?

4. Can you think of another method that Eupalinos could have used to align the tunnel?

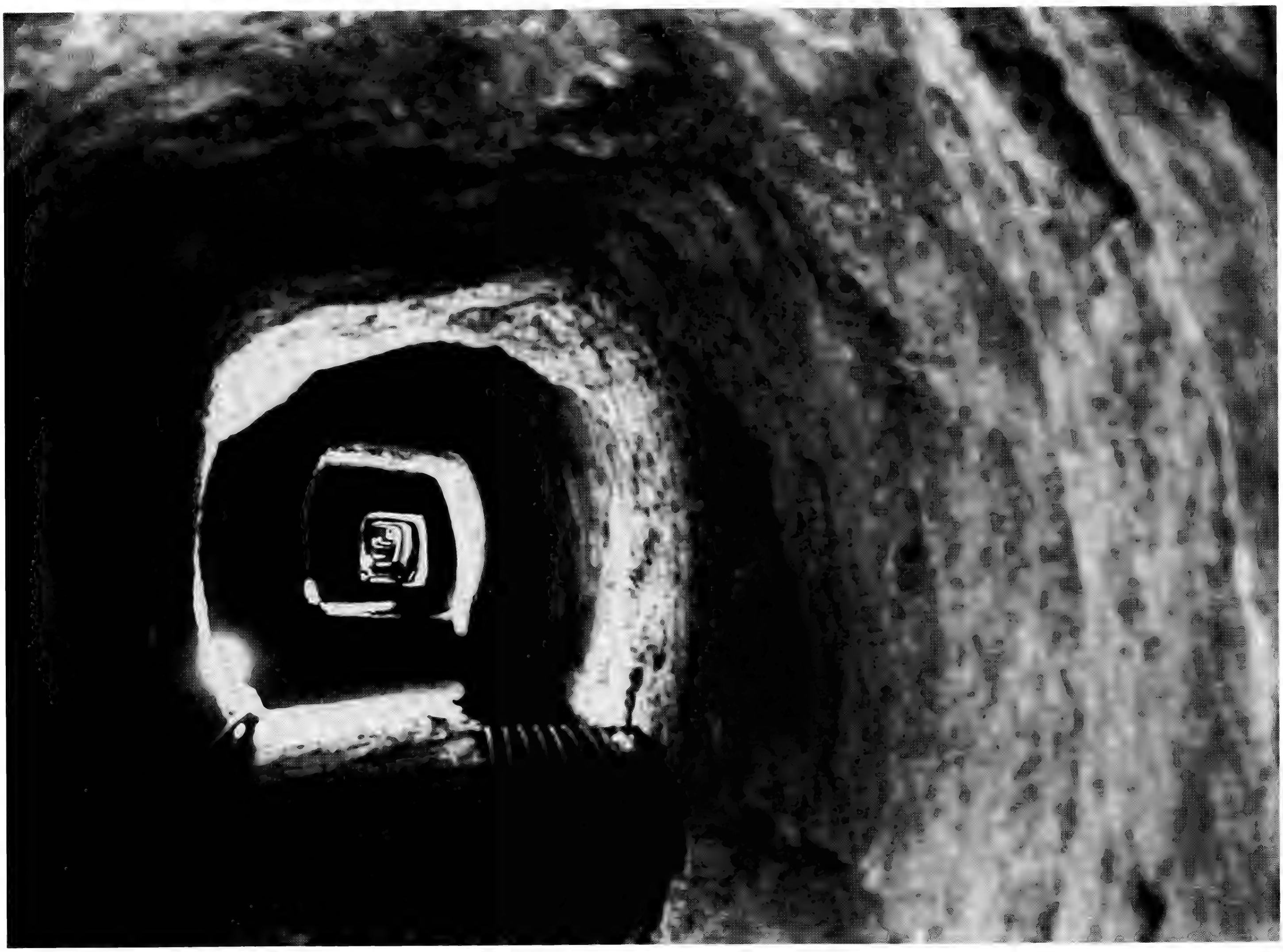


Figure 13. Tunnel interior, looking north. Modern grillwork on the floor covers the water channel.

About 30 meters inside the south entrance the narrow channel leaves the tunnel and heads east as an underground conduit leading to the ancient city. Its depth below the surface is much greater than that of the corresponding conduit coming in from the north. Inspection shafts from the conduit to the surface help trace its path for some distance, but the terminal point inside the city has not yet been found. It's possible that the conduit did not terminate at a single reservoir but branched out to supply several different locations.

To visit the tunnel today from the southern entrance, one first enters a small stone building (built in 1883) manned by a guard. A narrow rectangular opening in the floor contains a steep flight of wooden steps leading down into a walled passageway about twelve meters long, slightly curved, and barely wide enough for one person to walk through. The sides of the passageway are built of stone blocks joined without mortar. It is capped by a gabled roof formed by pairs of huge flat stones leaning against each other in a manner characteristic of ancient Greek construction. This passageway, shown in Figure 14, and a similar one at the northern entrance, were obviously constructed after the tunnel and water channel were completely excavated. Excess rubble from the excavation, which must have been considerable, was used to cover these passageways and camouflage them for protection against enemies or unwelcome intruders.

Figure 14. View through the narrow passageway looking south toward the entrance.

7. *Later history of the tunnel*

For centuries, the tunnel kept its secret. Its *existence* was known, but for a long time its exact whereabouts remained undiscovered. The earliest direct reference appears in the works of Herodotus a full century after construction was completed. Artifacts found in the tunnel indicate that it had been entered by the Romans, and there is a small shrine near the center from the Byzantine era (*ca.* 500-900 A.D.). At this point water constantly seeps through the roof and trickles down the walls. Perhaps this water was credited with miraculous properties, accounting for the shrine, which is made of several marble columns and thin white marble slabs decorated with Byzantine ornamentation.

It was from Samos that Byzantine Emperor Nikephoros Phokas embarked on a successful expedition in 960 to liberate Crete from the Arabs. This period marks the rapid decline of Samos. After the Turks invaded the island in 1453 it was depopulated until the seventeenth century, when it was re-occupied and ruled by an archbishop. During this period knowledge of the tunnel all but disappeared. The Russians occupied Samos briefly from 1772 to 1774. In the 1821 Greek uprising against the Turks, the Samians fought so valiantly that "to go to Samos" became a common expression meaning "to go to one's death." Although the Samians won a number of victories, the island was restored to the Turks in 1834 after the end of the war. However, Samos was given special privileges that amounted to autonomy. It was governed by a Greek prince chosen from the Christians on the island, and it had an elected house of representatives. In 1912 Samos was reunited with Greece.

In 1853 a man named Guérin uncovered the upper end of the subterranean conduit and excavated part of it, but stopped before reaching the tunnel itself. Abbot Kirillos from a nearby monastery later discovered the north entrance and persuaded Prince Constantine, the hereditary ruler of the island, to excavate it. Fifty men went to work in 1882 clearing and restoring about half the tunnel. They also restored the entire northern underground conduit and a portion of the southern conduit. And, on the foundation of an ancient structure, they built a small stone house that today marks the southern entrance.

In 1883, under the sponsorship of the German archaelogical institute in Athens, the archaeologist Ernst Fabricius surveyed the tunnel and published an excellent description, including a topographic sketch like that in Figure 15. The tunnel was again neglected for nearly a century until the Greek government cleared the southern half, covered the channel with protective grillwork, and installed electric lights so tourists could visit it safely. Hermann Kienast of the German archaeological institute in Athens has prepared a lengthy report describing more details about the construction of the tunnel, but this report was not published at the time this booklet was completed (early 1995).

The tunnel is one of the principal tourist attractions on Samos. A paved road from nearby Pythagorion leads to the stone building at the southern entrance. But many visitors who enter this building go no further because they are intimidated by the steep and narrow unlighted wooden staircase leading to the lower depths.

A portion of the northern part has also been cleared, and modern stone walls have been erected leading to the narrow staircase and passageway at the north entrance. Visitors can reach the northern entrance by walking around Mount Castro from the other side. As of 1994 no electric lights were installed in the northern portion of the tunnel.

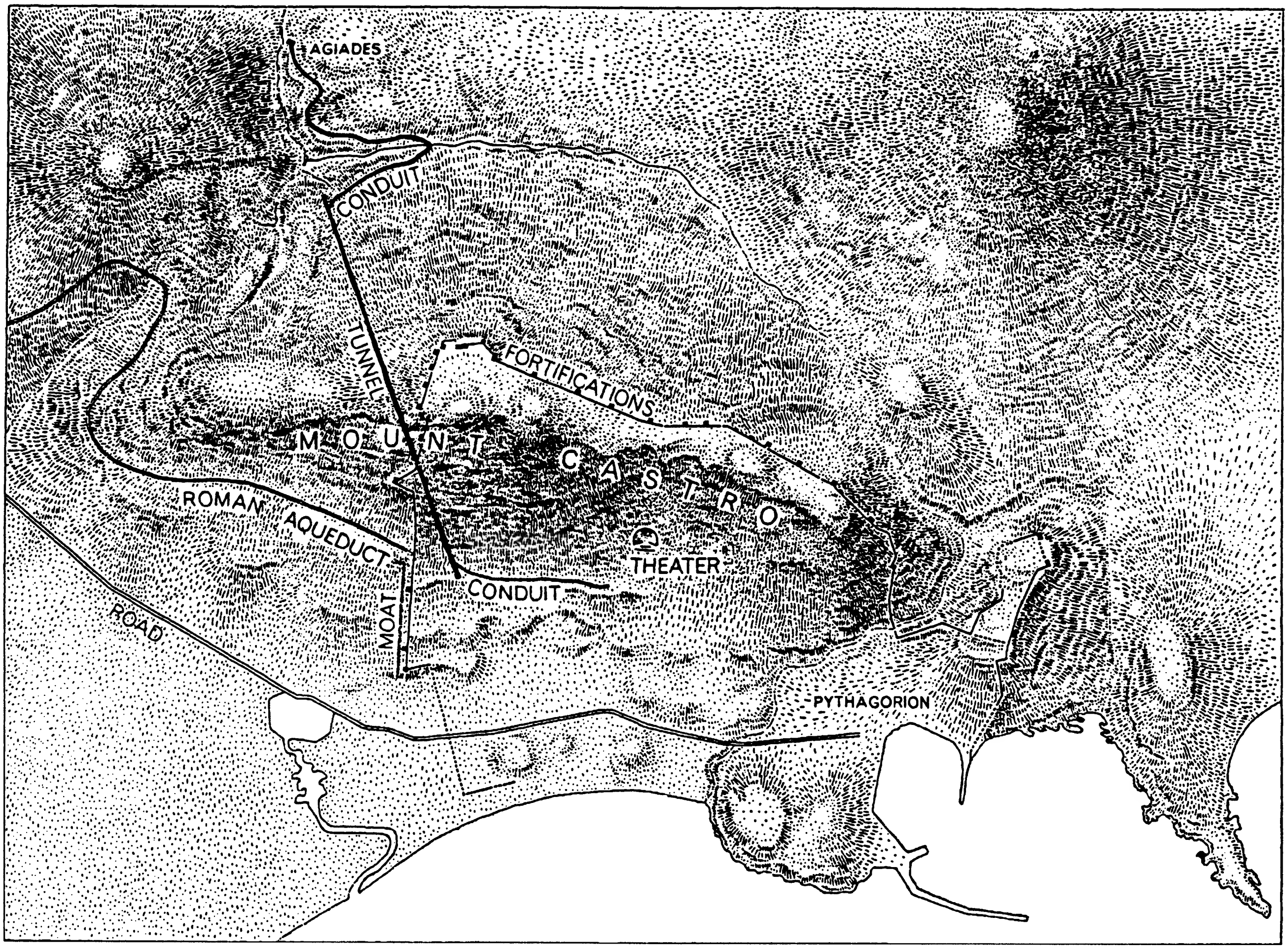


Figure 15. Topographic sketch of ancient Samos, adapted from a drawing prepared by Fabricius in 1884.

The historian Herodotus was born about 484 B.C. at Halicarnassus and travelled extensively in Greece, Macedon, Thrace, Persia, and Palestine. He lived in Samos in 457 and went to Athens about 447. Herodotus tells us that the Samos tunnel was only one of three remarkable engineering achievements on this island. The Greeks on Samos also built an impressive sea wall still in use today, 2500 years later, to protect the harbor. And, a few miles to the west, they constructed a magnificent temple to the goddess Hera. It was extolled by Herodotus as the largest he had ever seen. It was, in fact, one of the largest shrines of the ancient world, supported by 150 columns, each more than twenty meters tall. It was later surpassed by one of the seven wonders of the ancient world, the temple to Artemis at Ephesus in Asia Minor opposite Samos. Today, the temple of Hera lies in ruins, with only a single column standing at the site, a silent reminder of the spirit and ingenuity of the ancient Greeks.

8. *Recap*

Mathematics is a human endeavor whose history cannot be detached from the general history of human culture. Although mathematics is, by and large, a purely intellectual activity, its development has been intimately related to its applications and to the growth of technology.

This module tells the story of one of the earliest applications of mathematics to real life: determining the direction for excavating the Tunnel of Samos in the sixth century B.C., straight through the heart of a mountain. This program describes two methods that have been proposed to determine the direction for tunneling, using two crews digging toward each other.

The first method was suggested by Hero of Alexandria, five centuries after the tunnel was completed. It calls for a series of right-angled traverses around the mountain beginning at one entrance of the proposed tunnel and ending at the other, maintaining a constant elevation. By measuring the net distance traveled in each of two perpendicular directions, the lengths of two legs of a right triangle are determined, and the hypotenuse of the triangle is the proposed line of the tunnel. By laying out smaller similar right triangles at each entrance, markers can be then used by each crew to determine the direction for tunneling.

The principal mathematical idea involved in this particular method is the following:

The angles in a right triangle do not change if its two perpendicular legs are expanded or contracted by the same scaling factor.

This statement, which is intuitively obvious, can also be deduced as part of the theory of similar triangles in Euclidean geometry. For centuries before geometry was developed as a deductive system, Babylonian and Egyptian architects, engineers and astronomers undoubtedly used this principle as a working rule of thumb. Thales, a Greek mathematician and philosopher from Miletus, could have learned this and other empirical rules related to land measurement from his travels to Egypt in the sixth century B.C. In an attempt to fit these rules of thumb into a logically connected system, Thales began the Greek tradition of using logical reasoning to deduce properties of geometric figures. He is often credited with constructing a framework for geometry that became the foundation for the deductive system organized and expounded so well two centuries later by Euclid in his *Elements*. The particular property of similar triangles mentioned above could easily have found its way from Miletus to nearby Samos.

A second method, proposed more recently by two British historians of science, involves going over the top of the mountain along a direct path joining the two entrances. Reasons are given for believing that this method by itself is not accurate enough to ensure that the two entrances would be at the same elevation, and it is suggested that Eupalinos may well have used a combination of the two methods.

The Tunnel of Samos was neither the first nor the last to be excavated from both ends. This module mentions other examples, including the much shorter Tunnel of Hezekiah excavated near Jerusalem around 700 B.C., and the much longer tunnel under the English Channel completed in 1994. The Tunnel of Hezekiah required no mathematics at all; the Tunnel of Samos used very little mathematics: similar triangles, parallel lines, addition and subtraction of straight line distances; whereas the Channel Tunnel used the full power of modern technology which, in turn, is based in large part on mathematics developed over the last 2,500 years.

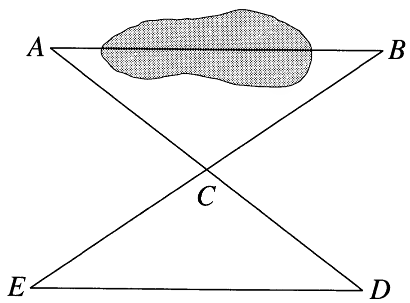
This module illustrates that mathematics is the offspring of both intellectual curiosity and the need to answer questions about the world in which we live.

Appendix: EXERCISES ON SIMILAR TRIANGLES (Selected from the workbook on *Similarity*)

Exercises 1 through 4 use similar triangles to determine the length of a line segment that cannot be measured directly because all or part of the segment is inaccessible.

1. In this exercise, A and B are separated by a pond, and we wish to determine the distance AB. In this situation, both points A and B are accessible from a third point C not on the line AB.

(a) Extend the line segment AC to the point D, making $CD = AC$ (in length). Extend segment BC to the point E, making $CE = BC$, as shown in the figure. Use congruent triangles to show that $AB = ED$.



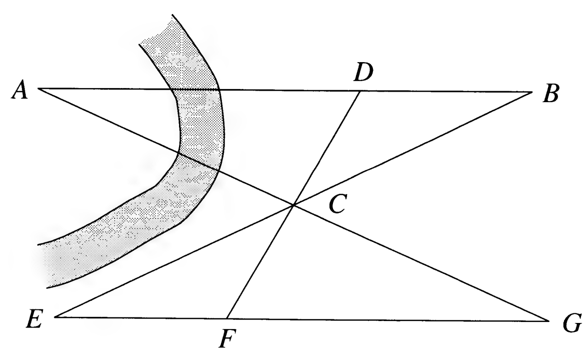
Exercise 1. Endpoints A and B accessible from C.

(b) Refer to part (a), but extend the line segment AC until CD has half the length of AC, and extend segment BC until CE has half the length of BC. Use similar triangles to show that $AB = 2ED$.

2. In this exercise we wish to determine distance AB when A and B are separated by a river, with both A and B visible from a third point C not on AB, but with only one endpoint B accessible from C.

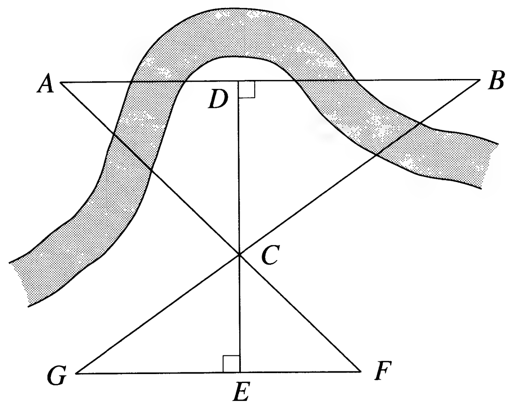
(a) Extend the line segment BC until $CE = BC$. On segment AB, choose a convenient point D accessible from C and extend DC until $CF = DC$. Let G be the point of intersection of the lines through AC and EF, as shown in the figure. Show that $AB = EG$.

(b) Refer to part (a), but extend segment BC until CE has one-third the length of BC, and extend DC until CF has one-third the length of DC. Show that $AB = 3EG$.



Exercise 2. Only one endpoint B is accessible from C.

3. In this example, both points A and B are visible from a third point C not on the line AB. Neither point is accessible from C, but a line through C perpendicular to segment AB intersects AB at a point D which is accessible from C. Extend the line segment DC until $CE = sDC$, where s is a convenient scaling factor. Draw a line through E perpendicular to segment CE. On this line, let F be the point collinear with A and C, and let G be the point collinear with B and C. Show that $AB = FG/s$.



Exercise 3. Both endpoints A, B are visible but not accessible from C.

4. In this example, the entire line segment AB is inaccessible, but the segment can be extended beyond A and B to two points D and E that are accessible from a third point C not on the line containing points A and B. Explain how to determine the length of the segment AB.

ADDITIONAL EXERCISES ON SIMILAR TRIANGLES

5. The edges of a triangle have lengths 3, 5 and 7. Find the lengths of the edges of a similar triangle whose perimeter is 42.

6. The altitude from the right angle to the hypotenuse of a right triangle divides the hypotenuse into segments of length 4 and 6. How long is the altitude? How long is it if each segment has length 6?

7. The altitude from the right angle to the hypotenuse of a right triangle divides the hypotenuse into segments of unequal length. Find the ratio of the longer segment to the shorter, given that one leg of the right triangle is three times as long as the other.

8. Erica, who is 1.7 meters tall, compares shadow lengths to find the height of a flagpole. She walks along the shadow of the pole 12 meters away from its base and finds that the tip of her shadow exactly matches the shadow of the top of the pole. Her sister Emily measures the horizontal distance from Erica to the end of the shadows and finds it to be 6.2 meters. Calculate the height of the flagpole.

9. A 20 ft pole and a 30 ft pole, each perpendicular to the ground, are a certain distance apart. Two cables (straight line segments) are used to connect the top of each pole to the bottom of the other. Show that the cables intersect at a point whose height above the ground does not depend on the distance between the poles.

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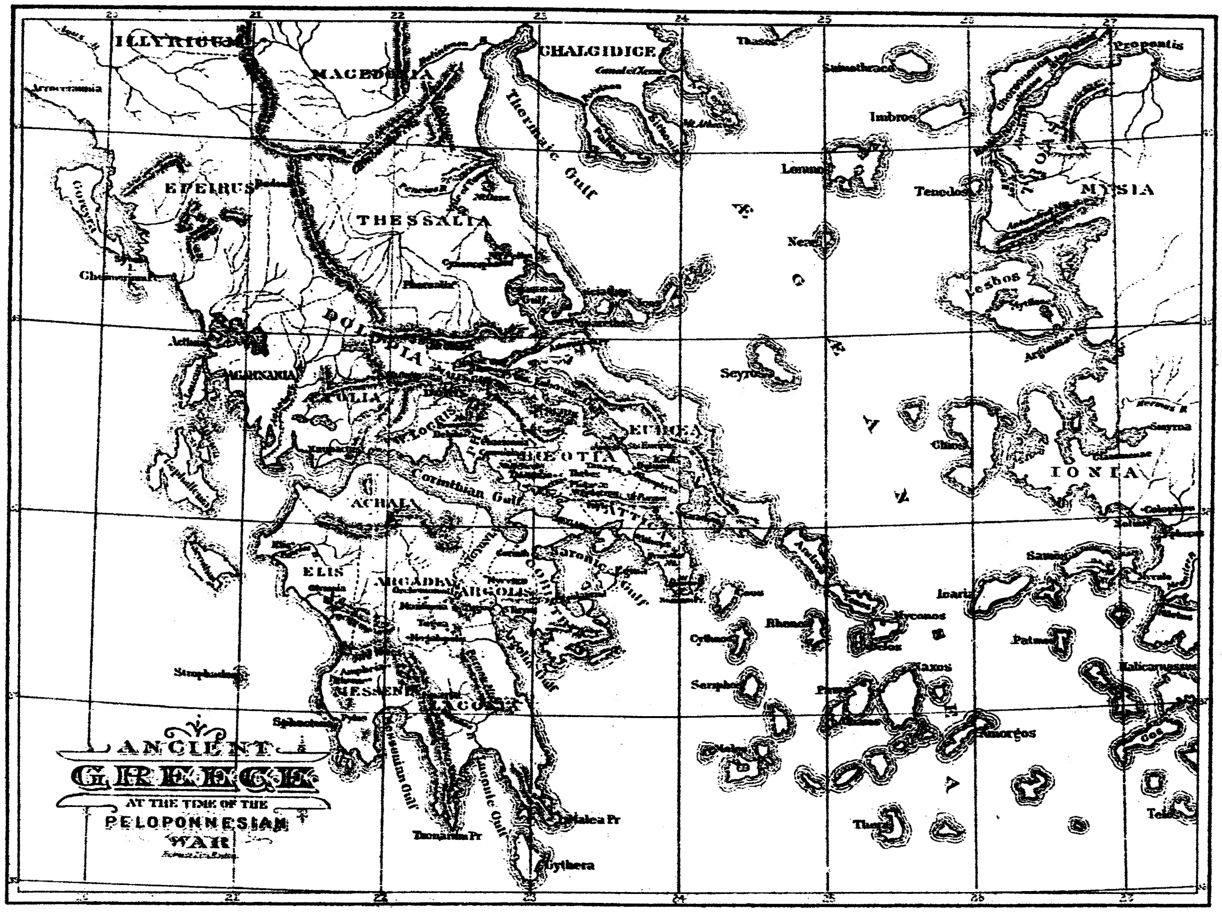

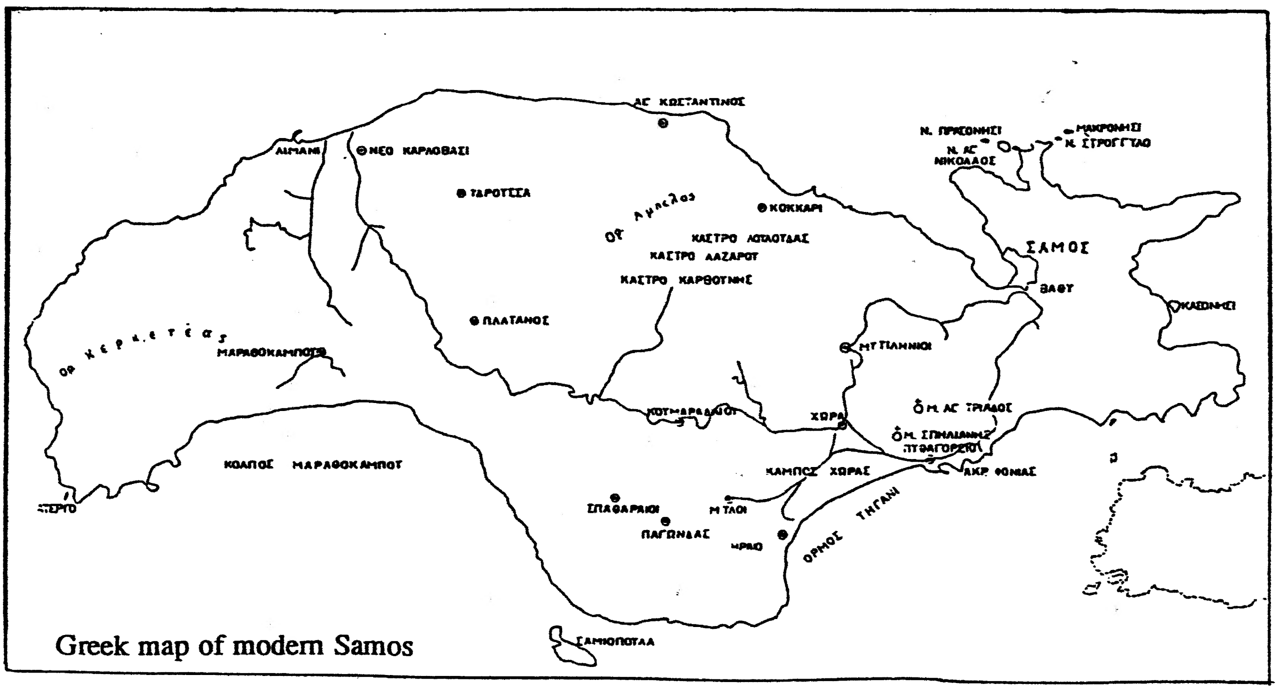

Greek map of modern Samos

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